Definitions and key facts for section 1.4

If A is an $m \times n$ matrix, with columns $\mathbf{a_1}, \ldots, \mathbf{a_n}$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then **the product of** A **and** $\mathbf{x}$, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = \begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots x_n\mathbf{a_n}.$$

Row-vector rule for computing $A\mathbf{x}$
If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Fact: Propeties of the matrix product $A\mathbf{x}$
Let A be a $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$, and c be a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

Fact: Matrix equations and linear systems
If A is an $m \times n$ matrix, with columns $\mathbf{a_1}, \ldots, \mathbf{a_n}$ and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \mathbf{b}$$

which in turn has the same solution set as the linear system with augmented matrix

$$\begin{bmatrix}\mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} & \mathbf{b}\end{bmatrix}.$$

A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ **spans** $\mathbb{R}^m$ if every vector $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_p}$, equivalently, $\text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_p}\} = \mathbb{R}^m$.

Fact: Consistency of $A\mathbf{x} = \mathbf{b}$ **for *all*** $\mathbf{b}$
For a given $m \times n$ matrix A, the following are equivalent. That is, either every statment is true, or every statement is false.

1. For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.